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2010 J. Phys. A: Math. Theor. 43 025002

(http://iopscience.iop.org/1751-8121/43/2/025002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **43** (2010) 025002 (12pp) doi:10.1088/1751-8113/43/2/025002

The effect of boundaries on the spectrum of a one-dimensional random mass Dirac Hamiltonian

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Received 14 September 2009, in final form 9 November 2009
Published 10 December 2009
Online at stacks.iop.org/JPhysA/43/025002

Abstract
The average density of states (DoS) of the one-dimensional Dirac Hamiltonian with a random mass on a finite interval $[0, L]$ is derived. Our method relies on the eigenvalue distributions (extreme value statistics problem) which are obtained explicitly. The well-known Dyson singularity $\overline{\varrho(\epsilon; L)} \sim -\frac{L}{|\epsilon| \ln^3 |\epsilon|}$ is recovered above the crossover energy $\epsilon_c \sim \exp -\sqrt{L}$. Below ϵ_c we find a log-normal suppression of the average DoS $\overline{\varrho(\epsilon; L)} \sim \frac{1}{|\epsilon|\sqrt{L}} \exp\left(-\frac{1}{L}\ln^2|\epsilon|\right)$.

PACS numbers: 72.15.Rn, 73.20.Fz, 02.50.−r

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Let us consider the one-dimensional (1D) Dirac equation $\mathcal{H}_\mathrm{D}\psi = \epsilon\psi$ for the Hamiltonian

$$\mathcal{H}_\mathrm{D} = -\alpha_x\,\mathrm{i}\partial_x + \beta\,\phi(x), \tag{1}$$

acting on a two-component spinor. $\phi(x)$ plays the role of a mass and will be taken as random. The Dirac matrices are chosen as $\beta = \gamma^0 = \sigma_1$ and $\alpha_x = \gamma^0\gamma^1 = -\sigma_2$, where σ_i are the usual Pauli matrices, corresponding to the representation of the Clifford algebra $\gamma^0 = \sigma_1$ and $\gamma^1 = -\mathrm{i}\sigma_3$. One-dimensional random Dirac Hamiltonians appear in several contexts of condensed matter physics ranging from disordered half-filled metals [1, 2], random spin-chain models (random antiferromagnetic spin-1/2 chains [3–5], random transverse field Ising spin-1/2 chains [6–8], spin-Peierls chains and spin-ladders [9, 10]) to organic conductors [11]. Another common representation of Dirac matrices $\beta = \sigma_1$ and $\alpha_x = \sigma_3$, related to the one chosen in the present paper by a unitary transformation $\mathcal{U} = (1 + \mathrm{i}\sigma_1)/\sqrt{2}$, shows that the Dirac equation has the form of the linearized Bogoliubov–de Gennes equation describing a superconductor with random gap [12, 13]. We obtain a connection to another important, and well-studied problem by squaring the Dirac Hamiltonian:

$\mathcal{H}_\mathrm{D}^2 = -\partial_x^2 + \phi(x)^2 + \mathrm{i}\gamma^1\phi'(x) = -\partial_x^2 + \phi(x)^2 + \sigma_3\phi'(x)$. This leads to a couple of isospectral supersymmetric Hamiltonians $H_\pm = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \phi(x)^2 \pm \phi'(x)$. The corresponding Schrödinger equation can be transformed into a Fokker–Planck equation $\partial_t P = \partial_x(\partial_x \mp 2\phi(x))P$ describing classical diffusion in random force field, studied in numerous works [8, 14–20]. If the random mass is a Gaussian white noise with characteristics $\overline{\phi(x)} = \mu\, g$ and $\overline{\phi(x)\phi(x')} - \overline{\phi}^{\,2} = g\,\delta(x-x')$, where $\overline{\cdots}$ denotes the average with respect to its realizations, an exact analytical expression for the density of states (DoS) was obtained [12, 17] (see section 8.2 of [13])[4]. Another solvable case is the one where the mass is chosen as a telegraph noise [22] (i.e. with exponentially decaying correlations). The same low-energy properties are obtained. This is related to the fact that any short-range correlated noise, upon large-scale renormalization, reduces to a Gaussian white noise.

Here we focus on the case of Gaussian white noise with $\overline{\phi(x)} = 0$, which is known to yield a Dyson singularity for the DoS $\varrho(\epsilon \to 0) \simeq \frac{g}{|\epsilon|\ln^3(g/|\epsilon|)}$ and a delocalization transition at $\epsilon \to 0$. The aim of the present paper is to study how the Dyson singularity of the DoS is affected by boundary conditions. We denote by $\varrho(\epsilon; L)$ the DoS of the Hamiltonian (1) on a finite interval $[0, L]$. We will obtain the behaviour of the average DoS $\overline{\varrho(\epsilon; L)}$ for a finite length.

2. Boundary conditions

Let us specify the two different types of boundary conditions which we shall consider.

Type (D). A simple way to introduce boundary conditions ensuring confinement in a domain $\mathcal{D}$ is to consider the Dirac equation $[\mathrm{i}\partial\!\!\!/ - \phi(x)]\psi(x,t) = 0$ with infinite mass $\phi(x) = \Phi \to \infty$ for x outside $\mathcal{D}$. This leads to the so-called bag model of hadronic physics [23]: the boundary conditions are $(1 + \mathrm{i}\vec{n}\cdot\vec{\gamma})\psi|_{\partial\mathcal{D}} = 0$, where $\vec{n}$ is the unit vector normal at the boundary $\partial\mathcal{D}$ of the domain. They force the vanishing of the component of the current density $\vec{\mathcal{J}}_\psi = \bar{\psi}\vec{\gamma}\psi$ perpendicular to the boundary, where $\bar{\psi} = \psi^\dagger\gamma^0$. In our case, for $x \in \mathbb{R}^-$ the stationary solution of the Dirac equation for a constant mass is a plane wave $\psi(x) = u\,\mathrm{e}^{\mathrm{i}kx}$ where the spinor u is the solution of $(\alpha_x k + \beta\Phi)u = \epsilon u$, for $\epsilon^2 = k^2 + \Phi^2$. In the limit $\Phi \to \infty$ it becomes an evanescent wave with $k \simeq -\mathrm{i}\Phi$; this yields $(1 - \mathrm{i}\gamma^1)\psi(0) = 0$. Similarly, at the other boundary we find $(1 + \mathrm{i}\gamma^1)\psi(L) = 0$. We denote these constraints as 'type (D) boundary conditions'. With our representation of the Clifford algebra, they read $(1-\sigma_3)\psi(0) = (1+\sigma_3)\psi(L) = 0$, i.e. each of the two components of the bispinor $\psi = (\varphi, \chi)$ must vanish at one side: $\chi(0) = \varphi(L) = 0$. This condition obviously ensures the absence of Dirac current flow $\mathcal{J}_\psi = -\psi^\dagger\sigma_2\psi$ across the boundaries.

Type (S). A second interesting choice is given by $(1 + \sigma_3)\psi(0) = (1 + \sigma_3)\psi(L) = 0$. These conditions coincide with Dirichlet boundary conditions for the associated supersymmetric Schrödinger Hamiltonian $H_\pm$ and will be denoted as 'type (S)'.[5]

[4] Note also [21] where the most general 1D Dirac Hamiltonian $\mathcal{H}_\mathrm{D} = \sigma_2\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \sigma_1\phi(x) + \sigma_3\lambda(x) + V(x)$ was studied for ϕ, λ and V Gaussian white noises. This study showed that there is no other DoS singularity than the one arising in the random mass case.

[5] *General boundary conditions.* It is possible to set up more general conditions forcing the absence of probability current flow at the boundaries: these are the set of conditions parametrized by some real number ϑ [24]: $(\mathrm{e}^{\mathrm{i}\vartheta\gamma^5} + \mathrm{i}\vec{n}\cdot\vec{\gamma})\psi|_{\partial\mathcal{D}} = 0$ where γ^5 is the chirality matrix, given by $\gamma^5 = \gamma^0\gamma^1$ in dimension $d = 1+1$. These general boundary conditions are obtained as follows: one considers the Dirac equation in a bounded domain $\mathcal{D}$. The boundary conditions can be written as $\mathrm{i}\vec{n}\cdot\vec{\gamma}\psi|_{\partial\mathcal{D}} = (A + B\gamma^5)\psi|_{\partial\mathcal{D}}$, where $\vec{n}$ is the unit vector normal to the domain, A and B are two complex numbers. The fact that $(\vec{n}\cdot\vec{\gamma})^2 = -1$ leads to $A^2 - B^2 = 1$. Upon imposing the normal component of the Dirac current to vanish $\vec{n}\cdot\vec{\mathcal{J}}_\psi|_{\partial\mathcal{D}} = 0$, we find $A \in \mathbb{R}$ and $B \in \mathrm{i}\mathbb{R}$, and thus the desired form. With the representation chosen in our paper these general boundary conditions take the form $(\mathrm{e}^{-\mathrm{i}\vartheta_\pm\sigma_2} \pm \sigma_3)\psi(x_\pm) = 0$ with $x_- = 0$ and $x_+ = L$. For type (D) we have $\vartheta_+ = \vartheta_- = 0$, and for (S) $\vartheta_- = \pi$, $\vartheta_+ = 0$. Since $\vartheta_\pm \neq 0$ or π breaks the particle–hole symmetry, this case will not be considered here.

3. Eigenvalue distributions

We now consider the Dirac Hamiltonian (1) on $[0, L]$ for the two kinds of boundary conditions introduced above. The particle–hole symmetry takes the form $\psi \to \tilde{\psi} = \sigma_3\psi$: $\sigma_3\mathcal{H}_{\mathrm{D}}\sigma_3 = -\mathcal{H}_{\mathrm{D}}$. Since both boundary conditions preserve the particle–hole symmetry, eigenvalues appear in pairs $\pm\epsilon_n$ (by convention we choose $n = 1, 2, 3, \ldots$ and $\epsilon_n > 0$). We denote by $\varrho(\epsilon)$ the DoS per unit length for an *infinite* volume. It is related to the DoS of the finite-size system by $\varrho(\epsilon) = \lim_{L\to\infty} \varrho(\epsilon; L)/L$. Because of particle–hole symmetry we can restrict ourselves to $\epsilon > 0$.

Our method to obtain the average DoS $\overline{\varrho(\epsilon; L)}$ relies on the evaluation of the eigenvalue distributions $\mathcal{W}_n(\epsilon; L) \stackrel{\text{def}}{=} \overline{\delta(\epsilon - \epsilon_n[\phi, L])}$, derived in [25] for (S)-boundaries. Finding the distributions of the (ordered) variables ϵ_n corresponds to an 'extreme value problem' (here for correlated variables). We recall the idea of the method: let us imagine that we impose the boundary conditions to hold solely at one end of the interval, say $x = 0$. As x increases the two components of the spinor $\psi(x) = (\varphi(x), \chi(x))$, which solves the Dirac equation, vanish alternately; we denote by Λ_m the distance between a node of $\varphi(x)$ and the closest node of $\chi(x)$. Since the evolution of $\psi(x)$ is Markovian, the Λ_m's are identical and independently distributed (i.i.d.) random variables whose statistical properties are obtained by solving a first passage time problem (see [25] for details). Let us introduce the distribution $\pi_M(y)$ of the sum of M i.i.d. rescaled lengths $y = (\Lambda_1 + \cdots + \Lambda_M)\mathcal{N}(\epsilon)$, where

$$\mathcal{N}(\epsilon) = \int_0^\epsilon \mathrm{d}\epsilon'\,\varrho(\epsilon') \underset{\epsilon\to 0}{=} \frac{g/2}{[\ln(2g/\epsilon) - \mathrm{C}]^2 + \pi^2/4} + O\left(\frac{\epsilon^2}{\ln^2\epsilon}\right) \approx \frac{g}{2\ln^2(g/\epsilon)} \tag{2}$$

is the integrated DoS per unit length for an infinite volume; $\mathrm{C} = 0.577\ldots$ is the Euler–Mascheroni constant (the exact expression for $\mathcal{N}(\epsilon)$ may be found in [12, 13, 17]). This rescaling is motivated by the fact that $\overline{\Lambda_m} = 1/(2\mathcal{N}(\epsilon))$ [25], and hence $\overline{y} = M/2$. Both boundary conditions at $x = 0, L$ are satisfied whenever the sum $\Lambda_1 + \cdots + \Lambda_M$ coincides with the length L. Therefore, the distribution $\mathcal{W}_n(\epsilon; L)$ is given by[6]

$$\mathcal{W}_n(\epsilon; L) = L\varrho(\epsilon)\varpi_n(L\mathcal{N}(\epsilon)), \tag{3}$$

where the dimensionless function $\varpi_n(y)$ is related to the distributions $\pi_M(y)$ as follows:

Type (S). The same component of the Dirac spinor must vanish at the two sides of the interval. Therefore, L must coincide with a sum of an *even* number of lengths Λ_m's (see equation (132) of [25]):

$$\boxed{\varpi_n^{(S)}(y) = \pi_{2n}(y)}\,. \tag{4}$$

Type (D). Both spinor components must vanish at one side of the interval. Hence, L must coincide with a sum of an *odd* number of lengths Λ_m's:

$$\boxed{\varpi_n^{(D)}(y) = \pi_{2n-1}(y)}\,. \tag{5}$$

The distributions $\pi_n(y)$ are explicitly known in the low-energy limit $\epsilon \ll g$, i.e. for $gL \gg 1$, where $\mathcal{W}_n(\epsilon; L)$ is essentially concentrated below g. Indeed, the characteristic function of the lengths Λ_m's is given by [25] $\overline{\mathrm{e}^{-\alpha\Lambda}} \simeq \cosh^{-1}\sqrt{\alpha/\mathcal{N}(\epsilon)}$. Whence we may write

$$\pi_n(y) \simeq \int_{\mathcal{B}} \frac{\mathrm{d}q}{2\mathrm{i}\pi}\,\frac{\mathrm{e}^{qy}}{\cosh^n\sqrt{q}}, \tag{6}$$

[6] Note that the change of variable from ϵ to $L\mathcal{N}(\epsilon)$ corresponds to the so-called spectrum unfolding leading to a unit density of variables.

where the integration is taken along the Bromwich contour. Starting from this integral representation we obtain the following explicit formulae:

$$\boxed{\pi_{2k+1}(y) = \frac{2^{2k}}{\sqrt{\pi}\, y^{3/2}} \sum_{n=0}^{\infty} (-1)^{n+k} (2n+1) \binom{n+k}{n-k} \mathrm{e}^{-\frac{(n+1/2)^2}{y}}} \tag{7a}$$

$$\boxed{\pi_{2k}(y) = \frac{2^{2k}}{\sqrt{\pi}\, y^{3/2}} \sum_{n=0}^{\infty} (-1)^{n+k} n \binom{n+k-1}{n-k} \mathrm{e}^{-n^2/y}}, \tag{7b}$$

whose demonstration is provided in appendix A.

4. DoS on a finite interval

The average DoS can be related to the distributions recalled above:

$$\overline{\varrho(\epsilon; L)} = \sum_{n=1}^{\infty} \mathcal{W}_n(\epsilon; L). \tag{8}$$

Summation over n leads to

$$\boxed{\overline{\varrho(\epsilon; L)} = L\, \varrho(\epsilon)\, \mathcal{D}(L\mathcal{N}(\epsilon))}, \tag{9}$$

where the dimensionless function $\mathcal{D}(y)$ depends on boundary conditions. Note that this DoS has some interest for studying the problem of 1D classical diffusion in a random force field with dilute absorbers [26] (see also [27] for an analysis of this problem with the real space renormalization group).

Boundary conditions of type (S). The summation of the (S)-type distributions (4) gives

$$\mathcal{D}_S(y) = \int_{\mathcal{B}} \frac{\mathrm{d}q}{2\mathrm{i}\pi} \frac{\mathrm{e}^{q\,y}}{\sinh^2\sqrt{q}}. \tag{10}$$

The integrand is meromorphic in the complex plane (there is no branch cut since $\sqrt{q}$ is the argument of an even function). The integral can be computed from the residue theorem. The integrand possesses a single pole at $q = 0$ with residue equal to unity and an infinite number of double poles on the real axis at $q = q_n = -(n\pi)^2$, with $n = 1, 2, 3, \ldots$. Using that $\sinh^2\sqrt{q} \underset{q\sim q_n}{\simeq} \frac{1}{4q_n}(q-q_n)^2$, we find the residues $\mathrm{Res}\big[\frac{\mathrm{e}^{qy}}{\sinh^2\sqrt{q}}\,;\, q_n\big] = \frac{\mathrm{d}}{\mathrm{d}q}\big[\frac{(q-q_n)^2\,\mathrm{e}^{qy}}{\sinh^2\sqrt{q}}\big]_{q=q_n} = (2+4q_n)\,\mathrm{e}^{q_n y}$. Therefore, we have

$$\mathcal{D}_S(y) = 1 + 2\sum_{n=1}^{\infty} (1 - 2(n\pi)^2 y)\, \mathrm{e}^{-(n\pi)^2 y} = \frac{4}{\sqrt{\pi}\, y^{3/2}} \sum_{n=1}^{\infty} n^2\, \mathrm{e}^{-n^2/y}, \tag{11}$$

where the second series expansion may readily be found from Poisson's summation formula (see appendix B). We may also check that the summation of (7*b*) leads to (11).

Boundary conditions of type (D). Compared to the (S) case, an additional $\cosh\sqrt{q}$ appears in the integrand upon summation of (5):

$$\mathcal{D}_D(y) = \int_{\mathcal{B}} \frac{\mathrm{d}q}{2\mathrm{i}\pi} \frac{\cosh\sqrt{q}}{\sinh^2\sqrt{q}}\, \mathrm{e}^{q\,y}. \tag{12}$$

It adds a sign $(-1)^n$ to the residues obtained in the previous case and we thus find

$$\mathcal{D}_D(y) = 1 + 2\sum_{n=1}^{\infty} (-1)^n (1 - 2(n\pi)^2 y)\, \mathrm{e}^{-(n\pi)^2 y} = \frac{1}{\sqrt{\pi}\, y^{3/2}} \sum_{n=0}^{\infty} (2n+1)^2\, \mathrm{e}^{-(2n+1)^2/4y}. \tag{13}$$

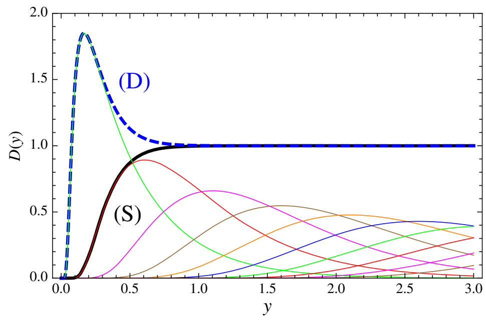

Figure 1. Functions $\mathcal{D}_S(y)$ (black continuous line) and $\mathcal{D}_D(y)$ (dashed blue line). First distributions $\varpi_n(y)$ are also plotted in thin lines.

The two different boundary conditions can be treated on the same footing by writing

$$\mathcal{D}_{S,D}(y) = \left(1 + 2y\frac{\mathrm{d}}{\mathrm{d}y}\right)\sum_{n\in\mathbb{Z}}(\pm 1)^n\,\mathrm{e}^{-(n\pi)^2 y} \tag{14}$$

and

$$\boxed{\mathcal{D}_{S,D}(y) = \frac{4}{\sqrt{\pi}\,y^{3/2}}\sum_{n=0}^{\infty}(n+\eta)^2\,\mathrm{e}^{-\frac{(n+\eta)^2}{y}}} \quad \text{with} \quad \eta = \begin{cases} 0 & \text{for } S \\ 1/2 & \text{for } D. \end{cases} \tag{15}$$

This second series expansion allows us to analyse the low-energy behaviour. It is worth noting that this representation for $\mathcal{D}(y)$ is very similar to the corresponding one for $\varpi_1(y)$, for both kind of boundary conditions. In the (S) case equations (A.4) and *rhs* of (13) only differ by the $(-1)^{n+1}$ in the sum, while in the (D) case (11) and *rhs* of (A.2) differ by a $(-1)^n(2n+1)$. As a result we have $\mathcal{D}(y \to 0) \simeq \varpi_1(y)$ in both cases. The two functions are plotted in figure 1. $\mathcal{D}_S(y)$ presents a monotonic behaviour which leads to a diminution of the low-energy DoS for small energies. Interestingly for (D) boundary conditions, while exponentially suppressed for $y \to 0$, $\mathcal{D}_D(y)$ increases for intermediate values of y; as a result (D) boundary conditions induce an increase of the DoS at intermediate energies.

The low-energy DoS presents a log-normal suppression:

$$\boxed{\overline{\varrho_S(\epsilon; L)} \simeq \frac{16}{|\epsilon|\sqrt{2\pi g L}}\,\mathrm{e}^{-\frac{2}{gL}\ln^2(g/|\epsilon|)} \qquad \text{for} \quad |\epsilon| \ll \epsilon_c = g\,\mathrm{e}^{-\sqrt{gL/2}}} \tag{16}$$

and

$$\boxed{\overline{\varrho_D(\epsilon; L)} \simeq \frac{4}{|\epsilon|\sqrt{2\pi g L}}\,\mathrm{e}^{-\frac{1}{2gL}\ln^2(g/|\epsilon|)} \qquad \text{for} \quad |\epsilon| \ll \epsilon_c' = g\,\mathrm{e}^{-\sqrt{2gL}}}\,. \tag{17}$$

The DoS reaches its maximal value at $\epsilon_* \approx g\mathrm{e}^{-gL/4}$ in the (S) case and at $\epsilon_*' \approx g\mathrm{e}^{-gL}$ in the (D) case.

The weaker log-normal suppression in the (D) case as compared to the (S) case is due to the fact that for a given realization of the disorder, the spectra for the two different kinds of boundary conditions are such that $\epsilon_1^D < \epsilon_1^S < \epsilon_2^D < \epsilon_2^S < \cdots$, where $\{\epsilon_n^{D,S}\}$ denotes the spectrum for boundary conditions of type (D) and (S) respectively.

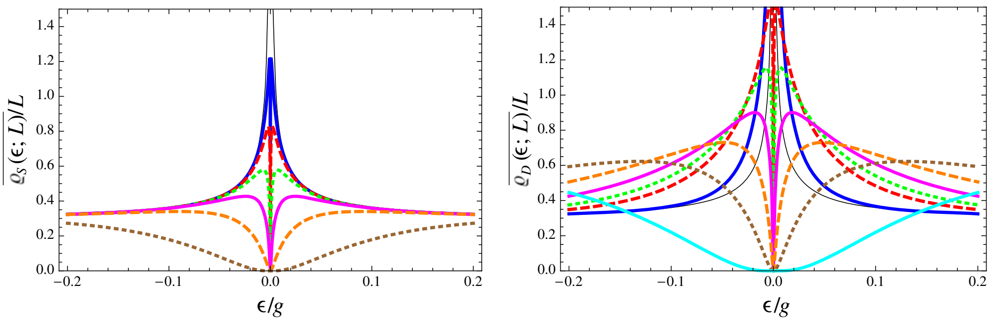

Figure 2. Average DoS per unit length for a finite L; parameter is $g = 1$. Left: (S) boundary conditions for sizes $L = \infty$, 30, 25, 20, 15, 10 and 5. Right: (D) boundary conditions for sizes $L = \infty$, 12, 6, 5, 4, 3, 2 and 1.

Since $\mathcal{D}(y \to \infty) \simeq 1$, equations (11, 13), we recover the Dyson singularity for intermediate energies, as expected:

$$\overline{\varrho(\epsilon;L)} \simeq L\,\varrho(\epsilon) \approx \frac{L\,g}{|\epsilon|\ln^3(g/|\epsilon|)} \qquad \text{for} \quad \epsilon_c \ll |\epsilon| \ll g. \tag{18}$$

At higher energies $|\epsilon| \gg g$, one should recover the free DoS: $\overline{\varrho(\epsilon;L)} \simeq L/\pi$.

The average DoS $\overline{\varrho(\epsilon;L)}$ is represented for various values of L in figure 2. For the lowest energies $\epsilon \lesssim \epsilon_* \sim g\mathrm{e}^{-gL}$, the DoS is suppressed in both cases. In the intermediate range $\epsilon_* \lesssim \epsilon \lesssim \epsilon_c \sim g\mathrm{e}^{-\sqrt{gL}}$, the effect of the boundaries is to reduce the DoS in the (S) case but surprisingly to increase the DoS in the (D) case.

As L is decreased the Dyson singularity is rapidly converted to a strong depletion of the low-energy DoS. For (S) boundary conditions this occurs for a surprisingly relatively large length $L_* \sim 10/g$. For the (D) case the memory of the Dyson singularity persists up to smaller lengths since an increase of the low-energy DoS is apparent up to $L'_* \sim 1/g$ (figure 2).

5. Conclusion

By using known results on the energy-level distributions $\mathcal{W}_n(\epsilon;L)$ we have obtained the average density of states of a random Dirac Hamiltonian on a bounded domain $[0, L]$. For $gL \gg 1$, we have seen that the DoS does not feel the boundary as energies become larger than the energy scale $\epsilon_c \sim g\,\mathrm{e}^{-\sqrt{gL}}$. In the intermediate range $\epsilon_* \lesssim \epsilon \lesssim \epsilon_c$, where $\epsilon_* \sim g\,\mathrm{e}^{-gL}$, the DoS may be diminished by boundaries, (S) case, or increased, (D) case. For the lowest energies the DoS presents a log-normal suppression for both kinds of boundary conditions.

Note that the effect of one boundary condition was already studied in [10], where the average local DoS on a semi-infinite line was computed by the Berezinskiĭ technique. These authors found an *increase* of the local DoS by a factor $\ln(g/|\epsilon|) \gg 1$ close to the boundary (at a distance much smaller than the scale $g^{-1}\ln^2(g/|\epsilon|)$). Here, by imposing boundary conditions at the two sides of finite interval, we have shown that boundary conditions can induce both an increase or a decrease of the DoS.

It is interesting to interpret our result within the classical diffusion problem. The (S) boundary corresponds to diffusion in a random force field with two absorbing boundaries while the (D) case corresponds to one reflecting boundary and one absorbing. The return probability is the Laplace transform of the DoS; therefore, the log-normal suppression of the average DoS coincides with a log-normal decay of the average return probability $\int_0^L \mathrm{d}x\, \overline{P(x,t\to\infty|x,0)} \sim \exp -\frac{1}{gL}\ln^2(g^2t)$.

To close this paper, let us examine the relation between our main result (9), (15) with the localization properties. It is only below the crossover energy $\epsilon_c \sim g\,\mathrm{e}^{-\sqrt{gL}}$ that eigenstates do feel the boundaries. The transition corresponds to the case where the localization length is of the order of the system size $\xi_{\epsilon_c} \sim L$; therefore, this criterion suggests that the eigenstate of energy ϵ is localized on a scale $\xi_\epsilon \sim g^{-1}\ln^2(g/|\epsilon|)$.

The question of localization was studied in several works ([28, 29] for a tight binding Hamiltonian with random hoppings and in [17] for the continuous supersymmetric Hamiltonian) that have demonstrated the vanishing of the Lyapunov exponent[7] at the band edge $\gamma(\epsilon\to 0) \sim g\,\ln^{-1}(g/|\epsilon|)$. The Lyapunov exponent provides another possible definition of the inverse localization length $\widetilde{\xi}_\epsilon \stackrel{\mathrm{def}}{=} 1/\gamma$, therefore, much shorter than the previous one: $\widetilde{\xi}_\epsilon \sim g^{-1}\ln(g/|\epsilon|) \ll \xi_\epsilon$. In other terms the Lyapunov exponent analysis would put the 'localization threshold'[8] at $\epsilon_* \sim g\mathrm{e}^{-gL}$, much below than $\epsilon_c \sim g\,\mathrm{e}^{-\sqrt{gL}}$.

The existence of two characteristic length scales was pointed out in several works [3, 5, 10, 17]. These authors have found that the average Green's function (i.e. the two point correlation function) decays over[9] $\xi_\epsilon \sim g^{-1}\ln^2(g/|\epsilon|)$. The existence of two length scales was interpreted by Fisher [3] as a consequence of fluctuations (see also [5]). The 'typical' localization length $\widetilde{\xi}_\epsilon$ characterizes the decay of a typical wavefunction (more precisely $\widetilde{\xi}_\epsilon$ is related to the average of the *logarithm* of the wavefunction) and the 'average' localization length ξ_ϵ controls the decay of the *average* correlation functions. Since we have considered an *average* quantity $\overline{\varrho(\epsilon;L)}$, the fact that we have extracted the scale ξ_ϵ is consistent with Fisher's argument.

Several arguments support the existence of a delocalization transition at $\epsilon\to 0$. Yet they are of quite different nature and it is interesting to provide a brief review. The first four arguments are bulk properties of the model. (i) The Lyapunov exponent vanishes at low energies: $\gamma(\epsilon) = 1/\widetilde{\xi}_\epsilon \to 0$ for $\epsilon\to 0$ [17]. (ii) The calculation of the average Green's function [5, 10, 17]. (iii) The dc conductivity of the model was computed at the Dirac point in [1, 2] and was found to be finite[10]. (iv) The statistical properties of the zero mode wavefunction [9, 33, 34] indicate long-range power law correlations. Another two arguments are obtained from scattering analysis. (v) The distribution of the transmission probability through a finite slab of length L at zero energy was obtained in [35]; in particular the average transmission decreases like $1/\sqrt{L}$, that is slower than the behaviour $1/L$ for a quasi-1D weakly-disordered conducting wire. (vi) The time delay distribution presents a log-normal distribution at zero energy [35, 36]: $P_L(\tau\to\infty) \sim \frac{1}{\tau\sqrt{L}}\exp -\frac{1}{8gL}\ln^2(g\tau)$, with moments diverging with the

[7] We recall that the Lyapunov exponent characterizes the exponential growth rate of the envelope of the wavefunction [13, 30] $\gamma \stackrel{\mathrm{def}}{=} \lim_{x\to\infty}\frac{\Xi(x)}{x}$, where Ξ controls the envelope of the wavefunction $\psi(x) = \mathrm{e}^{\Xi(x)}\times$ oscillations.

[8] Note that in the scaling theory of localization [31] the notion of a localization threshold (or mobility edge) designates the energy separating localized and delocalized states for the *infinite* system. Within this usual terminology the model we are studying is always in a localized phase apart strictly at $\epsilon = 0$. What we denote here by 'localization threshold' is the energy separating localized and delocalized states for a *finite* system of size L. With this definition there cannot be a sharp localization transition, due to fluctuations over disorder realizations.

[9] Note that a related quantity first obtained, [15, 16], is the propagator, the inverse Laplace transform of the Green's function, or more precisely the average conditional probability $\overline{P(x,t|x',0)} = \phi_0(x)\phi_0(x')^{-1}\langle x|\mathrm{e}^{-tH_+}|x'\rangle$ of the related Fokker–Planck equation, where $\phi_0(x)$ is the zero mode of the supersymmetric Hamiltonian H_+; the result of these works was reproduced in [32] from stochastic Riccati analysis and in [19] with the real space renormalization group.

[10] In the localized phase in 1 dimension, the ac conductivity vanishes at small frequency according to Mott's law $\mathrm{Re}\,\sigma(\omega) \sim \omega^2\ln^2\omega$ [30].

length of the disordered region[11] [36]: $\overline{\tau(0)} = 2L$ and $\overline{\tau(0)^n} \sim g^{-n}\mathrm{e}^{2n^2 gL}$. Finally, two arguments consider the problem on a finite interval taking into account boundary conditions. (vii) The study of extreme value statistics of energy levels reveals spectral correlations for $\epsilon \to 0$ [25], whereas a localized phase is characterized by the absence of level correlations [38]. (viii) Finally, the present work has shown the influence of the boundary on $\overline{\varrho(\epsilon \to 0; L)}$ due to delocalization.

It remains an interesting issue to go beyond this analysis of the localization characterized by the two lengths $\widetilde{\xi}_\epsilon$ and ξ_ϵ, and comprehend better the role of fluctuations, for example for the DoS $\varrho(\epsilon; L)$. Another challenging issue would be to extend our results to two dimensions. Since our method is specific to one-dimensional systems, such an extension would necessitate the development of other methods. This might be of interest due to the recent revival of two-dimensional random Dirac Hamiltonian physics, motivated by the study of graphene.

Acknowledgments

It is our pleasure to acknowledge Alain Comtet for stimulating discussions, and in particular for pointing out the relation discussed in appendix C. Moreover, we thank Pierre Le Doussal for discussions.

Appendix A. Extreme value spectral statistics

We give the first distributions introduced in the text. We compute the integral (6) using the residue theorem. This requires the expansion of the denominator near the pole $\kappa_n = -\frac{\pi^2}{4}(2n+1)^2, n \in \mathbb{N}$: $\cosh\sqrt{q} = \frac{(-1)^n}{\pi(2n+1)}(q-\kappa_n)\big[1+\frac{q-\kappa_n}{-4\kappa_n}+\frac{12+4\kappa_n}{6}\big(\frac{q-\kappa_n}{-4\kappa_n}\big)^2+\frac{10+4\kappa_n}{2}\big(\frac{q-\kappa_n}{-4\kappa_n}\big)^3+\cdots\big]$. Some algebra gives:

$$\pi_1(y) = \sum_{n=0}^{\infty}(-1)^n\pi(2n+1)\,\mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y} \tag{A.1}$$

$$= \frac{1}{\sqrt{\pi}\,y^{3/2}}\sum_{n=0}^{\infty}(-1)^n(2n+1)\,\mathrm{e}^{-(2n+1)^2/4y} \tag{A.2}$$

$$\pi_2(y) = \sum_{n=0}^{\infty}[\pi^2(2n+1)^2 y - 2]\,\mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y} \tag{A.3}$$

$$= \frac{4}{\sqrt{\pi}\,y^{3/2}}\sum_{n=0}^{\infty}(-1)^{n+1}n^2\,\mathrm{e}^{-n^2/y} \tag{A.4}$$

$$\pi_3(y) = \sum_{n=0}^{\infty}(-1)^n\pi(2n+1)\left[\frac{\pi^2(2n+1)^2}{2}y^2 - 3y + \frac{1}{2}\right]\mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y} \tag{A.5}$$

$$= \frac{1}{2\sqrt{\pi}\,y^{3/2}}\sum_{n=0}^{\infty}(-1)^{n+1}(2n+1)[(2n+1)^2 - 1]\,\mathrm{e}^{-(2n+1)^2/4y} \tag{A.6}$$

[11] This suggests that a particle of energy $\epsilon \approx 0$ injected in a disordered region of size L will spread over the full interval and remain trapped for a very long time depending on L, whereas in a strongly localized phase the particle would only explore a typical size $\widetilde{\xi}_\epsilon$ which is reflected in the fact that the time delay distribution reaches a limit distribution $\lim_{L\to\infty} P_L(\tau) = P_\infty(\tau)$ [37].

$$\pi_4(y) = \sum_{n=0}^{\infty} \left[\frac{\pi^4(2n+1)^4}{6} y^3 - 2\pi^2(2n+1)^2 y^2 + 2\left(\frac{\pi^2(2n+1)^2}{3} + 1\right) y - \frac{4}{3} \right] \mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y} \tag{A.7}$$

$$= \frac{8}{3\sqrt{\pi}\, y^{3/2}} \sum_{n=0}^{\infty} (-1)^n n^2 (n^2-1)\, \mathrm{e}^{-n^2/y} \tag{A.8}$$

(the distribution $\pi_1(y)$ was already obtained in [7] where the energy ϵ_1 is interpreted as the gap of a spin chain. It is also related to the distribution of the random bond lengths in the real space renormalization group procedure [3, 6, 19, 27]; $\pi_2(y)$ and $\pi_4(y)$ are given in [25]).

Expressions (A.2), (A.4), (A.6), (A.8) are obtained from equations (A.1), (A.3), (A.5), (A.7) by using the Poisson formulae given in the next appendix and $\pi_3(y) = \left(-2y^2 \frac{\mathrm{d}}{\mathrm{d}y} - 3y + \frac{1}{2}\right)\pi_1(y)$, $\pi_2(y) = -2\left(2y\frac{\mathrm{d}}{\mathrm{d}y} + 1\right) \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y}$ and $\pi_4(y) = \left(\frac{8}{3}y^3 \frac{\mathrm{d}^2}{\mathrm{d}y^2} + 8\left(y^2 - \frac{y}{3}\right)\frac{\mathrm{d}}{\mathrm{d}y} + 2y - \frac{4}{3}\right) \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{\pi^2}{4}(2n+1)^2 y}$.

A.1. Generating function

We propose now a method that allows for a systematic determination of the distributions $\pi_n(y)$. Let us introduce the generating function

$$\mathcal{G}(z, y) = \sum_{n=1}^{\infty} z^n \pi_n(y) = z \int_{\mathcal{B}} \frac{\mathrm{d}q}{2\mathrm{i}\pi} \frac{\mathrm{e}^{qy}}{\cosh\sqrt{q} - z}, \tag{A.9}$$

where summation was performed in the convergence radius $|z| < 1$. The function arccos is single valued in the convergence disc, therefore we can write the poles of the integrand as $q_n = -(n\pi + \arccos z)^2$ with $n \in \mathbb{N}$. We compute the residues by using $\frac{1}{2\sqrt{q_n}} \sinh\sqrt{q_n} = (-1)^n \frac{\sqrt{1-z^2}}{2(n\pi + \arccos z)}$ (it is helpful to note that $\arccos(x \pm \mathrm{i}0^+) = \arccos(x) \mp \mathrm{i}0^+$ for x real in $[-1, +1]$), whence

$$\mathcal{G}(z, y) = \frac{z}{y} \frac{\partial}{\partial z} \sum_{n=0}^{\infty} (-1)^n\, \mathrm{e}^{-(n\pi + \arccos z)^2 y}. \tag{A.10}$$

In order to apply the Poisson summation formula (B.1) and get the generating function of the distributions with odd indices, we exploit the symmetry with respect to a change in sign of the argument $\sum_{n=0}^{\infty} (-1)^n \mathrm{e}^{-(n\pi+\arccos z)^2 y} = -\sum_{n=-\infty}^{-1} (-1)^n \mathrm{e}^{-(n\pi+\arccos(-z))^2 y}$:

$$\mathcal{O}(z, y) = \frac{\mathcal{G}(z, y) - \mathcal{G}(-z, y)}{2} = \frac{z}{2y} \frac{\partial}{\partial z} \sum_{n=-\infty}^{+\infty} (-1)^n\, \mathrm{e}^{-(n\pi + \arccos z)^2 y}$$

$$= \frac{z}{y} \frac{\partial}{\partial z} \frac{1}{\sqrt{\pi y}} \sum_{n=0}^{\infty} T_{2n+1}(z)\, \mathrm{e}^{-\frac{(n+1/2)^2}{y}}. \tag{A.11}$$

Here the $T_n(z)$ are the Chebychev polynomials of the first kind $T_n(x) = \cos(n \arccos(x))$. They may be rewritten as

$$T_n(x) = \frac{n}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{n-2k}, \tag{A.12}$$

where $\lfloor x \rfloor$ is the integer part. Upon insertion into (A.11) we obtain

$$\frac{1}{z}\mathcal{O}(z,y) = \sum_{n=0}^{\infty} z^{2n}\,\pi_{2n+1}(y)$$
$$= \frac{1}{\sqrt{\pi}\,y^{3/2}}\sum_{n=0}^{\infty}(-1)^n(2n+1)\,\mathrm{e}^{-\frac{(n+1/2)^2}{y}}\sum_{k=0}^{n}(-1)^k\binom{n+k}{n-k}(2z)^{2k}. \tag{A.13}$$

Using that $\binom{n}{p} = 0$ for $p > n$, we may relax the constraint on the summation with respect to k and extract (7*a*). Note that equation (7*a*) allows to recover (A.2) and (A.6).

In order to evaluate $\pi_n(y)$ with $n = 2k$, recall that we deal with a distribution of n positive i.i.d. random variables. It follows that we may obtain $\pi_{2k}(y)$ by convolution of $\pi_{2k-1}(y)$ and $\pi_1(y)$: $\pi_{2k}(y) = \int_0^y \mathrm{d}x\,\pi_{2k-1}(x)\pi_1(y-x)$. The integration is fairly cumbersome but one may verify that it yields (7*b*). This completes the computation of $\pi_n(y)$ for all positive integers n as announced in the main text. One may verify that the summation of all $\pi_{2k}(y)$ (or $\pi_{2k+1}(y)$) yields the densities given by equation (15).

Appendix B. Two useful Poisson formulae

Let us start by recalling the well-known Poisson formula $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(2\pi n)$ for any function $f(x)$ defined on $\mathbb{R}$, with $\hat{f}(k) = \int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\mathrm{i}kx} f(x)$ its Fourier transform. Applying this formula we obtain

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi n\eta}\,\mathrm{e}^{-\pi^2(n+\alpha)^2 y} = \frac{1}{\sqrt{\pi y}}\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\mathrm{i}\pi(n-\eta)\alpha}\,\mathrm{e}^{-\frac{(n-\eta)^2}{y}}, \tag{B.1}$$

used for $\pi_n(y)$ with even indices (set $\eta = 0$ and $\alpha = 1/2$) and for the DoS (set $\eta = 0$ or $\eta = 1/2$ and $\alpha = 0$). For the distributions $\pi_n(y)$ with odd indices we need (with $\eta = \alpha = 1/2$)

$$\sum_{n}(n+\alpha)\,\mathrm{e}^{2\mathrm{i}\pi n\eta}\,\mathrm{e}^{-\pi^2(n+\alpha)^2 y} = \frac{1}{\mathrm{i}(\pi y)^{3/2}}\sum_{n}(n-\eta)\,\mathrm{e}^{2\mathrm{i}\pi(n-\eta)\alpha}\,\mathrm{e}^{-\frac{(n-\eta)^2}{y}}. \tag{B.2}$$

Appendix C. A probabilistic interpretation of the result for type (S) boundary conditions

It is worth pointing that the function $\mathcal{D}_S(y)$ can be interpreted as the integrated distribution of the maximum of a Brownian excursion. Let us denote by $(x(t),\ 0 \leqslant t \leqslant 1)$ such an excursion. We establish a relation between the distribution of the maximum of such an excursion and the function $\mathcal{D}_S(y)$.

C.1. Maximal height of a Brownian excursion

Let us consider the distribution of the maximum M of a Brownian excursion $(x(t),\ 0 \leqslant t \leqslant 1)$ (a Brownian bridge constraint to be positive). It can be written as a ratio of two path integrals:

$$\mathrm{Proba}\,[x(t) \leqslant M] = \lim_{x_0\to 0^+}\frac{\int_{x(0)=x_0}^{x(1)=x_0}\mathcal{D}x(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^1\mathrm{d}t\,\dot{x}^2}\prod_{t=0}^{1}\theta(x(t))\theta(M-x(t))}{\int_{x(0)=x_0}^{x(1)=x_0}\mathcal{D}x(t)\,\mathrm{e}^{-\frac{1}{2}\int_0^1\mathrm{d}t\,\dot{x}^2}\prod_{t=0}^{1}\theta(x(t))}, \tag{C.1}$$

where $x_0 > 0$ is a regulator. The ratio of path integrals may be rewritten as

$$\mathrm{Proba}\,[x(t) \leqslant M] = \lim_{x_0\to 0^+}\frac{\langle x_0|\,\mathrm{e}^{-H_1}|x_0\rangle}{\langle x_0|\,\mathrm{e}^{-H_0}|x_0\rangle}, \tag{C.2}$$

where $H_0 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ acts on functions defined on $\mathbb{R}^+$ satisfying Dirichlet boundary condition at $x = 0$ and $H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ acts on functions defined on $[0, M]$ with Dirichlet boundary conditions. It follows that [39, 40]

$$\mathrm{Proba}\,[x(t) \leqslant M] = \sqrt{\frac{2}{\pi}} \left(\frac{\pi}{M}\right)^3 \sum_{n=1}^{\infty} n^2\, \mathrm{e}^{-\frac{1}{2}\left(\frac{n\pi}{M}\right)^2}. \tag{C.3}$$

C.2. First-exit time

We consider a Brownian motion on $[0, 1/\sqrt{2}]$ with a reflecting boundary condition at $x = 1/\sqrt{2}$. We denote by τ the time at which $x(t)$ hits $x = 0$ for the first time, starting from $x = 1/\sqrt{2}$, and τ_x the time needed to reach $x = 0$, starting from x. $h(x, q) = \overline{\mathrm{e}^{-q\tau_x}}$ obeys the BFPE $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}h(x, q) = q\, h(x, q)$ with boundary conditions $h(0, q) = 1$ and $\partial_x h(x, q)|_{1/\sqrt{2}} = 0$ [41]. We easily find

$$h(x, q) = \frac{\cosh \sqrt{2q}(x - 1/\sqrt{2})}{\cosh \sqrt{q}}\,; \tag{C.4}$$

therefore, distribution of time τ is given by inverse Laplace transform of

$$\overline{\mathrm{e}^{-q\tau}} = \frac{1}{\cosh \sqrt{q}}. \tag{C.5}$$

We introduce the sum of n i.i.d such variables: $y = \tau_1 + \cdots + \tau_n$. The distribution of this variable was introduced in the text, $\pi_n(y) = \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}q}{2\mathrm{i}\pi} \frac{\mathrm{e}^{q\,y}}{\cosh^n \sqrt{q}}$, where we have shown that $\mathcal{D}_S(y) = \sum_{n=1}^{\infty} \pi_{2n}(y) = \frac{4}{\sqrt{\pi}\, y^{3/2}} \sum_{n=1}^{\infty} n^2\, \mathrm{e}^{-n^2/y}$. Comparing with (C.3) we seen that the sum of these distributions coincides with the cumulative distribution of the Brownian excursion[12]

$$\mathcal{D}_S(y) = \mathrm{Proba}[x(t) \leqslant M = \pi\sqrt{y/2}]. \tag{C.6}$$

It would be interesting to know whether this remark is purely accidental or not.

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